

**Inexact Convex Relaxations for AC Optimal Power Flow:
Mathematical Formulation**

Based on Venzke, Chatzivasileiadis, and Molzahn (2020)

I. NOTATION

\mathcal{N} : buses; $\mathcal{G} \subset \mathcal{N}$: generators; \mathcal{L} : lines (i, j) . $V_k \in \mathbb{C}$: bus voltage; $S_{Gk} \in \mathbb{C}$: generator power; $W_{ij} = V_i V_j^*$: voltage product matrix; $Y \in \mathbb{C}^{|\mathcal{N}| \times |\mathcal{N}|}$: admittance matrix; c_{k2}, c_{k1}, c_{k0} : cost coefficients.

II. AC OPTIMAL POWER FLOW

A. Objective

$$\min_{V, S_G} \sum_{k \in \mathcal{G}} \left[c_{k2} (\Re\{S_{Gk}\})^2 + c_{k1} \Re\{S_{Gk}\} + c_{k0} \right] \quad (1)$$

Active and reactive power at bus k :

$$P_{Gk} = \sum_{j \in \mathcal{N}} |V_k| |V_j| (G_{kj} \cos \theta_{kj} + B_{kj} \sin \theta_{kj}), \quad (2)$$

$$Q_{Gk} = \sum_{j \in \mathcal{N}} |V_k| |V_j| (G_{kj} \sin \theta_{kj} - B_{kj} \cos \theta_{kj}), \quad (3)$$

where $G_{kj} = \Re\{Y_{kj}\}$, $B_{kj} = \Im\{Y_{kj}\}$, $\theta_{kj} = \theta_k - \theta_j$.

B. Constraints

$$(V_k^{\min})^2 \leq V_k V_k^* \leq (V_k^{\max})^2, \quad \forall k \in \mathcal{N}, \quad (4)$$

$$S_{Gk}^{\min} \leq S_{Gk} \leq S_{Gk}^{\max}, \quad \forall k \in \mathcal{G}, \quad (5)$$

$$|S_{ij}| \leq S_{ij}^{\max}, \quad \forall (i, j) \in \mathcal{L}, \quad (6)$$

$$S_{Gk} - S_{Dk} = \sum_{(k,j) \in \mathcal{L}} S_{kj}, \quad \forall k \in \mathcal{N}, \quad (7)$$

$$S_{ij} = Y_{ij}^* W_{ii} - Y_{ij}^* W_{ij}, \quad \forall (i, j) \in \mathcal{L}, \quad (8)$$

$$-\theta_{ij}^{\max} \leq \angle(V_i V_j^*) \leq \theta_{ij}^{\max}, \quad \forall (i, j) \in \mathcal{L}. \quad (9)$$

C. Nonconvexity: Voltage Product

Setting $W_{ij} = V_i V_j^*$ and lifting yields linear constraints in W , but requires:

$$W \succeq 0, \quad \text{rank}(W) = 1. \quad (10)$$

The rank-1 constraint is the *sole* source of nonconvexity.

III. CONVEX RELAXATIONS

A. DC-OPF (Linear)

Under $|V_k| \approx 1$, $\cos \theta_{ij} \approx 1$, $\sin \theta_{ij} \approx \theta_{ij}$, reactive power neglected:

$$P_{ij} \approx b_{ij} (\theta_i - \theta_j). \quad (11)$$

B. SDP Relaxation

Drop $\text{rank}(W)=1$ from (10), retain:

$$W \succeq 0. \quad (12)$$

Exact iff $\text{rank}(W^*) = 1$; recover \mathbf{V}^* via $W^* = \mathbf{V}^* (\mathbf{V}^*)^H$.

C. QC Relaxation (SOCP)

In polar form $V_k = v_k \angle \theta_k$, introduce auxiliary variables:

$$vv_{ij} = v_i v_j, \quad cs_{ij} = v_i v_j \cos \theta_{ij}, \quad si_{ij} = v_i v_j \sin \theta_{ij}. \quad (13)$$

McCormick envelope for vv_{ij} :

$$vv_{ij} \geq v_i^L v_j + v_i v_j^L - v_i^L v_j^L, \quad (14)$$

$$vv_{ij} \geq v_i^U v_j + v_i v_j^U - v_i^U v_j^U, \quad (15)$$

$$vv_{ij} \leq v_i^U v_j + v_i v_j^L - v_i^U v_j^L, \quad (16)$$

$$vv_{ij} \leq v_i^L v_j + v_i v_j^U - v_i^L v_j^U, \quad (17)$$

$$vv_{ij} \leq \frac{1}{2}(v_i^2 + v_j^2). \quad (18)$$

Cosine envelope over $[\theta_{ij}^L, \theta_{ij}^U]$:

$$\cos \theta_{ij} \geq \cos \theta_{ij}^L + \frac{\cos \theta_{ij}^U - \cos \theta_{ij}^L}{\theta_{ij}^U - \theta_{ij}^L} (\theta_{ij} - \theta_{ij}^L), \quad (19)$$

$$\cos \theta_{ij} \leq 1 - \sin \theta_{ij}^L (\theta_{ij} - \theta_{ij}^L) + \cos \theta_{ij}^L - 1. \quad (20)$$

Sine envelope over $[\theta_{ij}^L, \theta_{ij}^U]$:

$$si_{ij} \geq vv_{ij} [\sin \theta_{ij}^L + \cos \theta_{ij}^L (\theta_{ij} - \theta_{ij}^L)], \quad (21)$$

$$si_{ij} \leq vv_{ij} [\sin \theta_{ij}^U + \cos \theta_{ij}^U (\theta_{ij} - \theta_{ij}^U)]. \quad (22)$$

SOC linking constraint:

$$\left\| \begin{bmatrix} cs_{ij} \\ si_{ij} \end{bmatrix} \right\|_2 \leq vv_{ij}, \quad |S_{ij}|^2 \leq W_{ii} l_{ij}. \quad (23)$$

Compact QC form (T, M, C, S : envelopes for square, bilinear, cosine, sine):

$$W_{kk} = \langle v_k^2 \rangle^T, \quad \forall k \in \mathcal{N}, \quad (24)$$

$$\Re\{W_{ij}\} = \langle \langle v_i v_j \rangle^M \langle \cos \theta_{ij} \rangle^C \rangle^M, \quad (25)$$

$$\Im\{W_{ij}\} = \langle \langle v_i v_j \rangle^M \langle \sin \theta_{ij} \rangle^S \rangle^M. \quad (26)$$

IV. INEXACTNESS METRICS

A. Optimality Gap

$$\Delta_{\text{gap}} = \left(1 - \frac{f_{\text{cost}}^{\text{relax}}}{f_{\text{cost}}^{\text{local}}} \right) \times 100\%. \quad (27)$$

B. Cumulative Constraint Violation

For $x \in \{P_G, Q_G, |V|, \theta_{ij}, S_{ij}\}$:

$$x_{\text{viol}} := \sum_{k \in \mathcal{X}} \frac{\max(x_k^{\text{AC-PF}} - x_k^{\text{max}}, x_k^{\text{min}} - x_k^{\text{AC-PF}}, 0)}{x_k^{\text{max}} - x_k^{\text{min}}} \times 100\%. \quad (28)$$

C. Average Distance to Local Optimum

$$x_{\text{dist}} := \frac{1}{|\mathcal{X}|} \sum_{k \in \mathcal{X}} \frac{|x_k^{\text{relax}} - x_k^{\text{opt}}|}{x_k^{\text{max}} - x_k^{\text{min}}} \times 100\%. \quad (29)$$

V. PENALIZATION METHODS (SDP)

Augment (1) to drive W toward rank-1:

$$f_{\text{pen}} = f_{\text{cost}} + \epsilon_{\text{pen}} \text{Tr}\{W\}, \quad (30)$$

$$f_{\text{pen}} = f_{\text{cost}} + \epsilon_{\text{pen}} \sum_{k \in \mathcal{G}} \Im\{S_{Gk}\}, \quad (31)$$

$$f_{\text{pen}} = f_{\text{cost}} + \epsilon_{\text{pen}} \sum_{(i,j) \in \mathcal{L}} |S_{ij} - S_{ji}|. \quad (32)$$