Labwork 7: Cones, SOC Constraints & Second-Order Cone Programming

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General Instructions:

- You may use standard optimization modeling languages such as `CVXPY` (in Python) or `JuMP` (in Julia) along with a solver like `Gurobi`. `Gurobi` has free academic license.
- For every plot, ensure axes are labeled properly and include legends where multiple curves are shown. I know we do not submit Lab works, but it is important to learn good practices.
- Task marked with **[Handwritten]** must be done using handwritten derivations. It is important to learn.
- I used Claude to make this, of course with a lot of brainstorming. You can use AI, for brainstorming not for zero-shot assignment solving.
- I would follow this pattern: read question → read relevant class notes/slides *to* clear doubts before starting to code via AI or Piazza Peers → start coding with agents → 100% clear in math and whatever you coded → Think what should be the output of your code and why → Get/or Not get that output you wished → Analysis
- You must ensure that you know what you are doing. Try it, you will feel good.

SOC Feasible Sets & Chebyshev Center (Warming Up the Cone)

Objective: Visualize feasible sets defined by second-order cone constraints in 2D and solve a basic SOCP to find the Chebyshev center of a polytope.

Task 1.1: Visualizing SOC Constraints in 2D: Consider $x = [x_1, x_2]^T \in \mathbb{R}^2$. For each of the following SOC constraints, plot the feasible region by sampling a dense grid over $[-5, 5]^2$ and shading the feasible points:

(a) $\|x\|_2 \leq 3$ (a ball / disk)

(b) $\left\| \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|_2 \leq [1 \;\; 1]\, x + 4$ (a general SOC constraint)

(c) The **intersection** of (a) and (b).

Produce a single figure with three subplots side-by-side.

[Handwritten] For constraint (b), show that when you square both sides you get a quadratic inequality of the form $x^T Q x + r^T x + s \leq 0$ together with a linear constraint. Write out Q, r, and s explicitly. Is Q positive semidefinite? Why does this not contradict the convexity of the SOC feasible set? (Refer to Lecture notes on SoCP)

Task 1.2: Special Cases Verification: In lecture we discussed two special cases of the SOC constraint $\|Ax + b\| \leq c^T x + d$:

(A) If $A = 0$: it reduces to a linear inequality (halfspace).

(B) If $c = 0$: it reduces to a slice of an ellipsoidal cone.

Verify computationally: For case (A), set $A = 0$, $b = [2, 1]^T$, $c = [1, -1]^T$, $d = 5$ and show the feasible set is a halfspace. For case (B), set $A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $b = [0, 0]^T$, $c = [0, 0]^T$, $d = 2$ and show the feasible set is an ellipse. Plot both.

Task 1.3: Chebyshev Center of a Polygon: Given a convex polygon defined by m halfplanes $a_i^T x \leq b_i$ for $i = 1, \ldots, m$, the *Chebyshev center* is the center of the largest inscribed ball.

- Show that finding the Chebyshev center can be written as:

$$\max_{x_c, R} \; R \quad \text{s.t.} \quad a_i^T x_c + R\|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m, \quad R \geq 0.$$

- Use the polygon with vertices $\{(0,0),\ (4,1),\ (5,4),\ (2,6),\ (-1,3)\}$. Compute the halfplane representation $a_i^T x \leq b_i$ from these vertices.
- Solve the Chebyshev center problem using CVXPY/JuMP. Report the center x_c^* and the radius R^*, and plot the polygon with the inscribed circle.

Think an explanation in 2–3 sentences: why is the Chebyshev center problem an SOCP and not just an LP? Where exactly does the SOC constraint appear in the formulation? Identify the A, b, c, d matrices/vectors from the general SOC form $\|Ax + b\| \leq c^T x + d$ for this problem.

A Side Note: This series of tweets should make us pause and think about what lies ahead—while we remain busy optimizing for **immediate gains.**

– Why can't we build things like this? Is it really just about not having GPUs or funding?

The real question I keep coming back to is this: do we have the capacity to use resources well? Do we have the ideas? Do we have the skills? And when I say "we," I mean both IIT students and professors.

If we cannot build these, if we cannot even aspire to build and work toward them, then, in my honest opinion, we have little reason to take pride in whatever we think IIT stands for.

`As the saying goes: if we are not at the table, we are on the menu.`

I would love to have a Piazza thread going on this, but hey! I am a Prof. who has not other work unlike you folks.

Cheers!