Practice Sheet: LP Bounding and Duality
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1. Estimating Lower and Upper Bounds

Problem 1: The Search for the "Tightest" Bound

Consider the following Primal LP (P):

$$\begin{aligned} \text{maximize} \quad & z = 4x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 10 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) **Lower Bounding:** Choose three different feasible points (x_1, x_2) and calculate z. Observe how z increases as you move towards the "boundary" of the feasible set.

(b) **Naive Upper Bounding:** By looking at the constraints, it is obvious that $x_1 \leq 4$ and $x_2 \leq 10$. Therefore, $z = 4x_1 + x_2 \leq 4(4) + 10 = 26$. This is a valid upper bound. Can we do better?

(c) **The Dual Approach:** Assign multipliers y_1 to the first constraint and y_2 to the second. We want to find $y_1, y_2 \geq 0$ such that:

$$\begin{aligned} 1y_1 + 1y_2 &\geq 4 \quad \text{(Coeff of } x_1) \\ 1y_1 + 0y_2 &\geq 1 \quad \text{(Coeff of } x_2) \end{aligned}$$

Calculate the upper bound $10y_1 + 4y_2$ for $(y_1, y_2) = (1, 3)$. Compare this result to your naive bound of 26.

> **The Insight**
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> This problem shows that we can find multiple upper bounds. The goal of the **Dual Problem** is to find the *minimum* possible upper bound, which eventually "squeezes" down to meet the maximum lower bound (p^*).

2. Dual Problem Formulation

Problem 2: Economic Dispatch Logic

In a power grid, we often minimize cost. Consider a 2-generator system meeting a demand of 500 MW:

$$\begin{aligned} \text{minimize} \quad & C = 20P_1 + 50P_2 \\ \text{subject to} \quad & P_1 + P_2 \geq 500 \quad \text{(Demand constraint)} \\ & P_1 \leq 400 \quad \text{(Capacity of Gen 1)} \\ & P_1, P_2 \geq 0 \end{aligned}$$

Task: Formulate the Dual of this problem.

> **The Insight**
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> Students will see that the Dual variable for the "Demand" constraint must be non-negative, while the Dual variable for the "Capacity" constraint (if written as $P_1 \leq 400$) will have a different sign convention depending on how they standardise the inequality. This maps directly to the concept of **Shadow Prices** in electricity markets.

Problem 3: Symmetry of the Mapping Table

Find the Dual of the following problem. It contains a mix of "$\geq$", "$\leq$", and "$=$" constraints, along with a "free" variable.

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\begin{aligned}
\text{maximize} \quad & z = 3x_1 - 2x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + x_2 + x_3 \leq 10 \\
& x_1 - x_2 \geq 2 \\
& 2x_1 + 3x_3 = 15 \\
& x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \in \mathbb{R} \text{ (free)}
\end{aligned}
$$

> **The Insight**
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> This is a comprehensive test of the Duality Table. It forces students to recognize that the **direction** of the primal inequality and the **sign** of the primal variable both dictate whether the dual constraint is an inequality or an equation.

Problem 4: The Duality Gap and Feasibility

Suppose a Primal problem (P) is found to be **Infeasible**.

(a) According to the properties of Duality, what are the two possible states for the Dual problem (D)?

(b) If you find a feasible point for the Dual with an objective value of $d = -\infty$, what does this tell you about the Primal?

> **The Insight**
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> This reinforces the theoretical "worst-case" results. It helps students understand that the Dual problem is not just a math trick, but a logical mirror that reflects the feasibility and boundedness of the Primal system.