EET 110 Algorithms for Power Grid

Spring 2025-26 — Power System Optimization

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Lecture 2: Linear Programs

Parts of the lecture are taken from Laurent Lessard's University of Wisconsin–Madison Course

Linear programs

- Review: linear algebra– some review in LP context
- Some Simple Geometrical intuitions
- Standard form for LPs
- Transformations to Standard Forms

Matrix basics

A matrix is an array of numbers. $A \in \mathbb{R}^{m \times n}$ means that: $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$

Two matrices can be multiplied if inner dimensions agree:

$$\underset{(m \times p)}{C} = \underset{(m \times n)}{A} \; \underset{(n \times p)}{B} \quad \text{where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Transpose: A^T swaps rows and columns. $(A^T)_{ij} = A_{ji}$.

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$

A vector is a column matrix $x \in \mathbb{R}^n$: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ Transpose: $x^T = [x_1 \dots x_n]$

Inner and Outer Products

- **Inner Product:** Produces a scalar.

$$x^T y = [x_1 \dots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

  Often written $x \cdot y$ or $\langle x, y \rangle$.

- **Outer Product:** Produces an $n \times n$ matrix.

$$xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [y_1 \dots y_n] = \begin{bmatrix} x_1 y_1 & \dots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \dots & x_n y_n \end{bmatrix}$$

Characterizing Linear and Affine Functions

- We define a function $f(x_1, \ldots, x_m)$ as **linear** if it can be represented as a pure weighted sum of its variables:

$$f(x) = \sum_{i=1}^{m} a_i x_i = a^T x$$

- If function includes a constant offset b (or intercept a_0), it is classified as **affine**:

$$f(x) = a^T x + b$$

Illustrative Examples:

1. $f(x, y) = 3x - y$ is **linear** in (x, y).
2. $f(x, y) = 2xy + 1$ is **affine** with respect to x or y individually, but not jointly.
3. $f(x, y) = x^2 + y^2$ is **neither** linear nor affine.

Terminology Note:
In some contexts, "linear" is used as a broad term to include affine functions.

Matrix Version of Linear & Affine

Multiple linear or affine functions together

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n + b_2 \\ \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + b_m \end{matrix} \implies \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

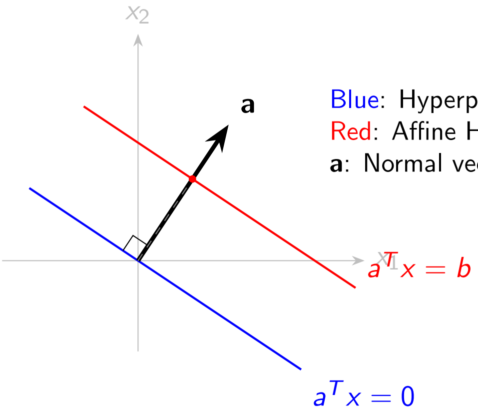
which can be written simply as $Ax + b$. Same definitions apply:

$$F \text{ is } \textbf{linear} \iff \exists A \in \mathbb{R}^{m \times n} \text{ s.t. } F(x) = Ax$$

$$F \text{ is } \textbf{affine} \iff \exists A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \text{ s.t. } F(x) = Ax + b$$

Geometry of Affine Equations

- The set $a^T x = 0$ is a **hyperplane** through the origin.
- The set $a^T x = b$ is an **affine hyperplane** (shifted).
- Vector a is **normal** to the hyperplane.



Blue: Hyperplane through origin
Red: Affine Hyperplane (shifted)
a: Normal vector

Linear and Affine Subspaces

- **Subspace:** Intersection of multiple hyperplanes The set of all points $x \in \mathbb{R}^n$ that satisfy a system of homogeneous linear equations:

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0 \quad \text{for } i = 1, \ldots, m \quad (\text{or } Ax = 0)$$

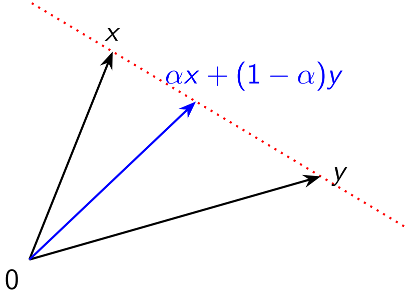
- **Affine Subspace:** When the right-hand side is non-zero ($Ax = b$), the solution set is called an **affine subspace**.
  - It maintains the same "flat" structure as a subspace but does not necessarily pass through the origin.
  - It can be viewed simply as a **translated (shifted) subspace**.

Affine combinations

If $x, y \in \mathbb{R}^n$, then the combination

$$w = \alpha x + (1-\alpha)y \quad \text{for some } \alpha \in \mathbb{R}$$

is called an **affine combination**.



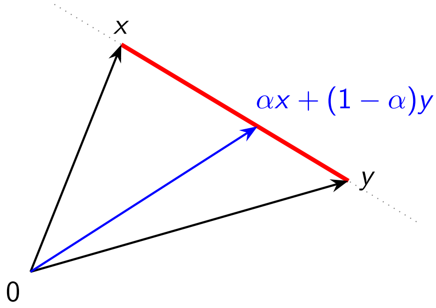
If $Ax = b$ and $Ay = b$, then $Aw = b$. So affine combinations of points in an (affine) subspace also belong to the subspace.

Convex combinations

If $x, y \in \mathbb{R}^n$, then the combination

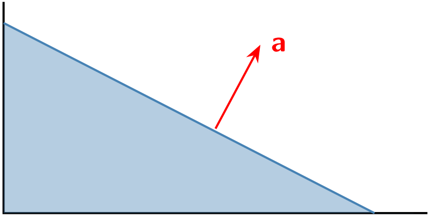
$$w = \alpha x + (1 - \alpha) y \quad \text{for some } 0 \leq \alpha \leq 1$$

is called a **convex combination** (for reasons we will learn later). It's the line segment that connects x and y.



Geometry of affine inequalities

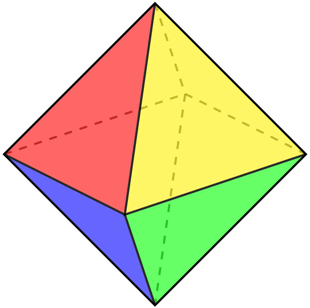
- A **halfspace** is the region on one side of a "flat wall" (hyperplane).
- It is defined by the inequality $\mathbf{a}^T\mathbf{x} \leq \mathbf{b}$
- Vector a is perpendicular to the boundary, and b determines the boundary's distance from the origin.



Halfspace

Geometry of affine inequalities

- A **polyhedron** (**polytope**) is defined as the subset of $\mathbb{R}^n$ formed by the intersection of multiple halfspaces, satisfying the system of linear inequalities $Ax \leq b$ (or $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$).



Convexity property: Consider a point $w = \alpha x + (1 - \alpha)y$ where $0 \leq \alpha \leq 1$. If the original points satisfy the constraints (i.e., $Ax \leq b$ and $Ay \leq b$), it follows that $Aw \leq b$.

Solutions of an LP

There are exactly three possible cases:

1. Model is **infeasible**:

2. Model is feasible, but **unbounded**:

3. Model has a solution which occurs **on the boundary** of the set.

The linear program

A linear program is an optimization model with:

- real-valued variables ($x \in \mathbb{R}^n$)
- affine objective function ($c^T x + d$), can be min or max.
- constraints may be:
  - affine equations ($Ax = b$)
  - affine inequalities ($Ax \leq b$ or $Ax \geq b$)
  - combinations of the above
- individual variables may have:
  - box constraints ($p \leq x_i$, or $x_i \leq q$, or $p \leq x_i \leq q$)
  - no constraints (x_i is unconstrained)

There are many equivalent ways to express the same LP

Standard Form

Every LP can be put into **Standard Form**:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Bat and Hockey Problem

$$\begin{aligned} \max_{x_1, x_2} \quad & 1200x_1 + 900x_2 \\ \text{s.t.} \quad & 4x_1 + 2x_2 \leq 4800 \\ & x_1 + x_2 \leq 1750 \\ & 0 \leq x_1 \leq 1000 \\ & 0 \leq x_2 \leq 1500 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \max_{x} \quad & \begin{bmatrix} 1200 \\ 900 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix} \\ & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \end{aligned}$$

This is in standard form, with:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}, \quad c = \begin{bmatrix} 1200 \\ 900 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transformation tricks

- converting min to max or vice versa (take the negative):

$$\min_x f(x) = -\max_x(-f(x))$$

Example: To minimize loss instead of maximizing profit:

$\max(1200x_1 + 900x_2) \iff \min(-1200x_1 - 900x_2)$

- reversing inequalities (flip the sign):

$$Ax \leq b \iff (-A)x \geq (-b)$$

Example: Willow Wood constraint:

$4x_1 + 2x_2 \leq 4800 \iff -4x_1 - 2x_2 \geq -4800$

- equalities to inequalities (double up):

$$f(x) = 0 \iff f(x) \geq 0 \quad \text{and} \quad f(x) \leq 0$$

Example: If Machine Hours were exactly 1750:

$x_1 + x_2 = 1750 \iff x_1 + x_2 \leq 1750$ and $x_1 + x_2 \geq 1750$

More Tricks

- inequalities to equalities (add slack):

$$f(x) \leq 0 \iff f(x) + s = 0 \quad \text{and} \quad s \geq 0$$

Example: Converting Willow Wood limit:

$4x_1 + 2x_2 \leq 4800 \iff 4x_1 + 2x_2 + s_1 = 4800, \; s_1 \geq 0$

- unbounded to bounded (add difference):

$$x \in \mathbb{R} \iff u \geq 0, \; v \geq 0, \quad \text{and} \quad x = u - v$$

Example: If x_1 (Bat Grips) could be negative (returns allowed):

Replace x_1 with $u_1 - v_1$ in objective: $1200(u_1 - v_1)\ldots$

More More tricks

- bounded to unbounded (convert to inequality):

$$p \le x \le q \iff \begin{bmatrix} 1 \\ -1 \end{bmatrix} x \le \begin{bmatrix} q \\ -p \end{bmatrix}$$

Example: Bat Grip Supply ($0 \le x_1 \le 1000$):
$\begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1 \le \begin{bmatrix} 1000 \\ 0 \end{bmatrix} \implies x_1 \le 1000$ and $-x_1 \le 0$

- bounded to nonnegative (shift the variable)

$$p \le x \le q \iff 0 \le (x - p) \quad \text{and} \quad (x - p) \le (q - p)$$

Example: If min production for x_1 was 100 (not 0):
$100 \le x_1 \le 1000 \iff 0 \le \hat{x}_1 \le 900$ (where $\hat{x}_1 = x_1 - 100$)