EEC 351 Linear Algebra Homework

Question 1: Matrix Operations and Properties

Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}.$$

Task

1. Compute AB and BA; discuss commutativity.
2. Calculate $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, and $\mathrm{tr}(AB)$; examine trace properties.

Question 2: Vector Norms and Distances

Given

$$\mathbf{x} = [3, -4, 0, 2]^T, \quad \mathbf{y} = [1, 2, -1, 3]^T$$

Task

1. Compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_\infty$, and cosine similarity $\cos\theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$.
2. For $\mathbf{x} \in \mathbb{R}^2$, interpret $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ geometrically.

Question 3: Eigenvalues & Eigenvectors

For covariance matrix

$$C = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix},$$

Task

1. Find eigenvalues and eigenvectors; orthogonalize and normalize them.

Question 4: Quadratic Forms and Definiteness

Let

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}, \quad f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}.$$

Task

1. Expand $f(\mathbf{x})$ in x_1, x_2.
2. Determine definiteness via eigenvalues.
3. ML connection: interpret definiteness in Hessian context and gradient descent stability.

Question 6:

Task

Let: $\sigma(z) = \frac{1}{1+e^{-z}}$ Compute and prove:

1. From first principles calculate $\sigma'(z)$. Deduce the well-known simplification $\sigma'(z) = \sigma(z)\,(1 - \sigma(z))$.

Question 7: PCA Optimization Proof

Let $X \in \mathbb{R}^{n\times d}$, covariance $C = X^T X$.

Task

Provide complete proofs for:

1. Positive Semi-definiteness: Show that C is positive semidefinite.
2. Unit-norm maximization leads to eigenvalue problem: Show that maximizing $v^T Cv$ subject to $\|v\| = 1$ via Lagrange multipliers gives

$$Cv = \lambda v,$$

   where λ is the corresponding Lagrange multiplier.
3. Characterizing the principal eigenvector: Argue that the solution v maximizing $v^T Cv$ corresponds to the **largest** eigenvalue of C.

References

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[4] S. Axler, *Linear Algebra Done Right.* Springer, 3 ed., 2015.

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[6] K. B. Petersen and M. S. Pedersen, "The matrix cookbook," 2012. Technical University of Denmark.

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